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Extended $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ characters, or modular properties of a simple non-rational conformal field theory*

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Abstract: We define extended $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ characters which include a sum over winding sectors. By embedding these characters into similarly extended characters of $N = 2$ algebras, we show that they have nice modular transformation properties. We calculate the modular matrices of this simple but non-trivial non-rational conformal field theory explicitly . As a result, we show that discrete $\mathrm{SL}(2,\mathbb{R})$ representations mix with continuous $\mathrm{SL}(2,\mathbb{R})$ representations under modular transformations in the coset conformal field theory. We comment upon the significance of our results for a general theory of non-rational conformal field theories.

Keywords: Conformal Field Models in String Theory, Black Holes in String Theory, Extended Supersymmetry.

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1. Introduction

A systematic theory of rational conformal field theories has been established. It includes modular transformation properties of characters, the construction of modular invariant partition functions, and, for example, a fairly systematic analysis of boundary conditions consistent with the chiral algebra. The general theory has proven extremely useful in constructing (supersymmetric) string theory compactifications [1] and their D-branes (apart from having applications in two-dimensional physical systems at criticality).

A similarly systematic study of non-rational conformal field theories is lacking. Recent progress in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ conformal field theory (see e.g. [2]) as well as in Liouville theory (see e.g. [3, 4]) and the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ conformal field theory indicates that such a theory is in reach. It would seem imperative then to analyze the features of simple non-rational conformal field theories that have proven instrumental in the theory of their rational cousins. One such feature is the modular transformation properties of the chiral characters of the conformal field theory. We study these transformation properties for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ conformal field theory in this paper. The $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ conformal field theory is perhaps the algebraically simplest of non-trivial non-rational conformal field theories, and it has importance as providing a black hole background in string theory [5]–[10].

In a larger framework, it is clear that a better understanding of non-rational conformal field theories should lead to a firmer grip on non-trivially curved non-compact string theory

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vacua, and therefore to a better understanding of the stringy theory of quantum gravity. Particular examples directly related to our paper include cosmological backgrounds and NS5-brane holography. Let us proceed then to lay bare some of the basic properties of a simple non-rational conformal field theory to make a small step in this direction.

We will restrict our computations to the case where the parent $\mathrm{SL}(2,\mathbb{R})$ conformal field theory is at integer level k. But our results are extendable at least to the case of generic rational k. Although the WZW $\mathrm{SL}(2,\mathbb{R})$ model (and hence the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset) is well defined for any real k, there are some settings were the case of integer k is singled out. In string theory the occurrence of integer level k appears in the context of double scaled little string theory, where it is the number of NS5-branes in the string background. A possible second instance where integer level k values are special is suggested by the embedding of $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ into non-minimal $N = 2$ algebras [11]. The $N = 2$ Landau-Ginzburg models with superpotentials given by the classified 14 singular polynomials of modality one [12], flow in the infrared to $N = 2$ superconformal field theories with $c = 3k/-2$, with integer k [11]. An algebraic description of these models in terms of modular invariants of non-minimal $N = 2$ theories would be a non-rational extension of the well-established equivalence [13] between $N = 2$ LG theories with superpotentials given by the ADE series of singular polynomials with modality 0 and the ADE modular invariants of minimal $N = 2$.

In section 2, we will define the extended characters of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ conformal field theory. We show in section 3 that the modular transformation properties of the continuous extended characters are derived with elementary means. The strategy we follow for the discrete characters is explained in section 4. In section 5 we remind the reader of the connection between the coset characters and characters for the $N = 2$ superconformal algebra. We will then put to use some known modular properties of the $N = 2$ superconformal algebra to derive the modular transformation properties of the extended discrete $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset characters. We thus construct the modular S- and T-matrices for this simple non-rational conformal field theory and we derive general lessons from this example in the concluding section.

2. The extended characters

In this section we define the extended characters for the bosonic coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. A motivation for defining extended characters is that the regular characters yield a continuous spectrum of $\mathrm{U}(1)$ charges under modular transformations and this problem is avoided for the extended characters.[1] We restrict to discrete and continuous representations of the parent $\mathrm{SL}(2,\mathbb{R})$ conformal field theory [14]. These include all normalizable (i.e. quadratically integrable) wavefunctions on the group manifold. We introduce the coset central charge:

$$c_{cs} = \frac{3k}{k-2} - 1 = 2 + \frac{6}{k-2}, \tag{2.1}$$

corresponding to a parent $\mathrm{SL}(2,\mathbb{R})$ algebra at level k.

[1] The charges of the coset are defined with respect to the $\mathrm{U}(1)$ global symmetry that remains after gauging the local $\mathrm{U}(1)$.

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Continuous representations. We label *continuous representations* of $\mathrm{SL}(2,\mathbb{R})$ by their Casimir $-j(j-1)$ determined by $j=\frac{1}{2}+is$ (where $s\in\mathbb{R}^+$) and their parity $\alpha\in\{0,\frac{1}{2}\}$. The characters of the coset can be obtained by expanding the $\mathrm{SL}(2,\mathbb{R})$ characters $\hat{\chi}_{j=1/2+is,\alpha}(q,z)$ into terms with definite U(1) charge $\alpha+r$ ($r\in\mathbb{Z}$), and expressing each such term as a character of a (time-like) U(1) boson times a character of the coset.[2] This yields

$$\begin{aligned}\hat{\chi}_{j=1/2+is,\alpha}(q,z) &= \mathrm{Tr}\, q^{L_0-(c_{cs}+1)/24}\, z^{J_0^3}\\ &= q^{\frac{s^2}{k-2}}\eta(\tau)^{-3}\sum_{r\in\mathbb{Z}} z^{\alpha+r}\\ &= \sum_{r\in\mathbb{Z}} z^{\alpha+r}\zeta_{r+\alpha}(\tau)\,\hat{\lambda}_{1/2+is,\alpha+r}(\tau)\,,\end{aligned} \tag{2.2}$$

with $q=e^{2\pi i\tau}$ and $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. The U(1) and the coset characters are:

$$\begin{aligned}\zeta_{r+\alpha}(\tau) &= q^{-\frac{(r+\alpha)^2}{k}}\eta(\tau)^{-1}\\ \hat{\lambda}_{1/2+is,\alpha+r}(\tau) &= \mathrm{Tr}\, q^{L_0-c_{cs}/24}\\ &= \eta(\tau)^{-2} q^{\frac{s^2}{k-2}} q^{\frac{(\alpha+r)^2}{k}}\,.\end{aligned} \tag{2.3}$$

It turns out to be natural to define new, extended characters (in the spirit of [21, 22]) that correspond to traces over direct sums of irreducible modules. In particular, for a given continuous representation, we define the extended characters by summing over all the coset modules whose U(1) charges differ by multiples of k.[3] While the coset characters $\hat{\lambda}_{1/2+is,\alpha+r}(\tau)$ transform in a complicated fashion under modular transformations, the extended characters have nice modular transformation properties, as we will see in the next sections.

We define the extended characters for the continuous representations as follows:

$$\begin{aligned}\hat{\Lambda}_{1/2+is,\alpha+r}(\tau) &= \sum_{n\in\mathbb{Z}}\hat{\lambda}_{1/2+is,\alpha+r+kn}(\tau)\\ &= \sum_{n\in\mathbb{Z}}\eta(\tau)^{-2} q^{\frac{s^2}{k-2}} q^{\frac{(\alpha+r+kn)^2}{k}}\\ &= \eta(\tau)^{-2} q^{\frac{s^2}{k-2}}\Theta_{2(\alpha+r),k}(\tau,0)\end{aligned} \tag{2.4}$$

[2]There are several methods to obtain the coset characters. The method decribed here is used in [15] (see also [16]). By embedding the parafermionic algebra into an $N=2$ algebra, a similar decomposition of the $N=2$ characters into coset and U(1) characters was obtained in [17]. We make use of this method in section 5. In [18] the characters were obtained by counting the multiplicities of the Hilbert space induced from $\mathrm{SL}(2,\mathbb{R})$ and in [19] by applying Felder's method to a Feigin-Fuchs representation of the parafermionic algebra. The coset characters have also been discussed in [20].

[3]This is reminiscent of similar techniques used for compact bosons at rational radius squared, and for characters of the $N=2$ superconformal algebra.

where the classical theta functions $\Theta_{m,k}(\tau,\nu)$ at level k are defined as:[4]

$$\Theta_{m,k}(\tau,\nu) = \sum_{p\in\mathbb{Z}+\frac{m}{2k}} q^{kp^2} z^{kp}\,, \qquad \text{with } z = e^{2i\pi\nu}\,. \tag{2.5}$$

Note that for a given s, there are $2k$ distinct extended continuous characters (labeled by $\alpha = 0, 1/2$ and $r = 0, 1\ldots, k-1$).

Discrete representations. We consider now *discrete lowest weight representations*[5] of $\mathrm{SL}(2,\mathbb{R})$ labeled by $2j \in \mathbb{Z}$. Only the representations with $0 < j < \frac{k}{2}$ give unitary representations in the coset theory.[6] The U(1) charges are $j+r$ ($r \in \mathbb{Z}$) The decomposition of the $\mathrm{SL}(2,\mathbb{R})$ discrete character $\chi_j(q,z)$ yields (see [15] for details):

$$\begin{aligned}
\chi_j(q,z) &= \mathrm{Tr}\, q^{L_0-(c_{cs}+1)/24}\, z^{J_0^3} \\
&= \frac{q^{-\frac{(j-\frac{1}{2})^2}{k-2}}\, z^{j-\frac{1}{2}}}{\vartheta_1(\tau,\nu)} \\
&= \sum_{r\in\mathbb{Z}} z^{j+r}\zeta_{j+r}(\tau)\,\lambda_{j,r}(\tau)
\end{aligned} \tag{2.6}$$

where[7]

$$\vartheta_1(q,z) = z^{-\frac{1}{2}} q^{\frac{3}{24}} \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}z)(1-q^n z^{-1})\,. \tag{2.7}$$

The coset characters are:

$$\lambda_{j,r}(\tau) = \eta(\tau)^{-2} q^{-\frac{(j-\frac{1}{2})^2}{k-2}} q^{\frac{(j+r)^2}{k}} S_r(\tau)\,, \tag{2.8}$$

where the function $S_r(\tau)$ for integer r is given by:

$$S_r(\tau) = \sum_{s=0}^{\infty}(-1)^s q^{\frac{1}{2}s(s+2r+1)}\,. \tag{2.9}$$

For the discrete representations we similarly define extended characters:

$$\begin{aligned}
\Lambda_{j,r}(\tau) &= \sum_{n\in\mathbb{Z}} \lambda_{j,kn+r}(\tau) \\
&= \eta(\tau)^{-2} q^{-\frac{(j-\frac{1}{2})^2}{k-2}} \sum_{n\in\mathbb{Z}} q^{\frac{(j+r+kn)^2}{k}} S_{r+kn}(\tau)\,.
\end{aligned} \tag{2.10}$$

We now turn to analyzing the modular transformation properties of these extended characters, which will occupy us for most of the rest of the paper.

[4] For rational k, $k = p/q$, we would introduce theta functions at level pq and sum over chiral momentum sectors that differ by p units. This is as in the case of a chiral boson at rational radius squared.

[5] The discrete representations of $\mathrm{SL}(2,\mathbb{R})$ are actually either highest weight or lowest weight, but both give the same modules for the coset theory.

[6] This restriction is further constrained to $\frac{1}{2} < j < \frac{k-1}{2}$ in the full conformal field theory [23, 14, 24].

[7] Both the expression for the discrete $\mathrm{SL}(2,\mathbb{R})$ characters in the second line of (2.6) *and* the decomposition into coset characters in the third line are valid only for $|q| < |z| < 1$.

3. Modular transformation of extended continuous characters

The continuous extended characters in eq. (2.4) are the product of three factors, each having good modular properties. For T-transformations ($\tau \to \tau + 1$), we use the formulas

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{i\pi}{12}}\eta(\tau) \\ \Theta_{m,k}(\tau+1,\nu) &= e^{\frac{i\pi m^2}{2k}}\Theta_{m,k}(\tau,\nu) \end{aligned} \tag{3.1}$$

to show that the following T-transformation holds:

$$\hat{\Lambda}_{1/2+is,\alpha+r}(\tau+1) = e^{2\pi i\left(\frac{s^2}{k-2}-\frac{1}{12}\right)}e^{2\pi i\frac{(\alpha+r)^2}{k}}\hat{\Lambda}_{1/2+is,\alpha+r}(\tau)\,. \tag{3.2}$$

Under S-transformations ($\tau \to -1/\tau$), using the identities:

$$\begin{aligned} \eta\left(-\frac{1}{\tau}\right) &= (-i\tau)^{1/2}\eta(\tau) \\ e^{-\frac{2\pi i}{\tau}\frac{s^2}{k-2}} &= \sqrt{\frac{-2i\tau}{k-2}}\int_{-\infty}^{+\infty}ds' e^{2\pi i\tau\frac{s'^2}{k-2}}e^{-\frac{4\pi iss'}{k-2}} \\ \Theta_{m,k}\left(-\frac{1}{\tau},\frac{\nu}{\tau}\right) &= \sqrt{\frac{-i\tau}{2k}}e^{\frac{\pi ik}{2\tau}\nu^2}\sum_{m'\in Z_{2k}}e^{-i\pi mm'/k}\Theta_{m',k}(\tau,\nu)\,. \end{aligned} \tag{3.3}$$

we find that

$$\begin{aligned} \hat{\Lambda}_{1/2+is,\alpha+r}\left(-\frac{1}{\tau}\right) &= \frac{2}{\sqrt{k(k-2)}}\int_0^{+\infty}ds'\cos\left(\frac{4\pi ss'}{k-2}\right)\times \\ &\quad\times\sum_{2\alpha'+2r'\in Z_{2k}}e^{-\frac{\pi i}{k}(2r'+2\alpha')(2r+2\alpha)}\hat{\Lambda}_{1/2+is',\alpha'+r'}(\tau)\,. \end{aligned} \tag{3.4}$$

Note that whatever the value of α is, the $2k$ characters with both $\alpha' = 0, 1/2$ appear in the righthand side of (3.4), with all the possible values of the continuous parameter s'.

We have thus derived the modular transformation properties of the extended continuous coset characters. It is clear that the continuous representations transform among themselves under modular T- and S-transformations. We can thus imagine building modular invariants using continuous representations only.

We can check, using the properties of the continuous and discrete Fourier transform that this part of the S-matrix squares to the charge conjugation matrix. Charge conjugation[8] is implemented for continuous representations by simply changing the sign of the U(1) charge associated to the coset character i.e. $\alpha + r \to -(\alpha + r)$.

We now turn to deriving the modular transformation properties for the extended discrete characters, which are less straightforward to compute.

[8]Although we haven't explicitly introduced a variable keeping track of the U(1) charge in our computation, it can easily be done.

4. Modular transformations of extended discrete coset characters: strategy

For the extended discrete coset characters, it is still possible to derive the transformation under the modular T-transformation $\tau \to \tau + 1$ in one step:

$$\Lambda_{j,r}(\tau+1) = e^{2\pi i(-\frac{(j-1/2)^2}{k-2}-\frac{1}{12})} e^{2\pi i \frac{(j+r)^2}{k}} \Lambda_{j,r}(\tau)\,, \tag{4.1}$$

where again we notice diagonality despite the sum over primaries with differing conformal weight.

To derive the transformation properties of the extended discrete coset characters under a modular S-transformation $\tau \to -\frac{1}{\tau}$, we will follow a more involved strategy. The idea underlying the computation is to embed the coset model into a model with a larger symmetry, to derive the transformation properties for the more symmetrical model, then to extract the modular transformation properties of the original extended coset characters. Recall that the modular properties of the string functions, i.e. of the characters of the compact and rational $\mathrm{SU}(2)/\mathrm{U}(1)$ coset, are straightforwardly derived from the corresponding transformation properties of the $\mathrm{SU}(2)$ parent Wess-Zumino-Witten conformal field theory (which has a larger symmetry) after decomposing the $\mathrm{SU}(2)$ WZW model with respect to a $\mathrm{U}(1)$ subgroup.

We choose to embed our model as a sub-sector of an $N = 2$ superconformal field theory. It is well known [11] that the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ operators occur as building blocks for the $N = 2$ superconformal algebra with $c > 3$. We will add a free boson to the coset model at the correct radius in order to enhance the symmetry of the theory to an $N = 2$ superconformal algebra. We then employ the character decomposition of $N = 2$ characters in terms of the compact boson characters and the bosonic $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset characters to link the modules of $N = 2$ to the desired coset modules. We then make use of known [25, 26] modular transformation properties of (extended) $N = 2$ characters (and of the compact boson characters) to derive the modular transformation properties of the constituent discrete bosonic coset characters.

That is a sketch of the strategy we choose, but one can think of at least two equivalent alternative strategies. The first would be to embed the coset into the original $\mathrm{SL}(2,\mathbb{R})$ theory,[9] but we would have to be careful with the negative weights of the time-like $\mathrm{U}(1)$.[10] The second would be to decompose directly the lemma we prove in appendix A in terms of $\mathrm{U}(1)$ charges and apply this directly to the bosonic coset characters.

Our strategy has the advantage that as a by-product, we will find useful links between extended $N = 2$ characters at $c > 3$ and the characters of our non-compact non-rational conformal field theory (which should be viewed as analogous to similar Gepner formulas [1] in the compact, minimal $N = 2$ case). In other words, our method lays bare some of the structure underlying computations in $N = 2$ models with $c > 3$.

[9] This has been done implicitly in [16].

[10] Care should also be taken in order not to violate the restriction mentioned in footnote 7.

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To implement our strategy, we first turn then to defining the relevant $N = 2$ characters in the next section, then to decompose them in section 5, next to recall their modular transformation properties in section 6. Finally, in section 7 we decompose these modular transformation properties in terms of the constituent compact boson and coset character.

5. Embedding $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ in $N = 2$

The current and primaries of unitary $N = 2, c > 3$ theories can by obtained from those of a *bosonic* $\mathrm{SL}(2,\mathbb{R})$ model by taking out the timelike boson associated to the bosonization of J^3 in $\mathrm{SL}(2,\mathbb{R})$, and introducing a spacelike boson to bosonize the $N = 2$ U(1) current. We refer the reader to [11] for the details of this construction. It is shown there that the continuous and discrete representations of $\mathrm{SL}(2,\mathbb{R})$ (with the unitarity bound $0 < j < k/2$ in the discrete), give rise to all the unitary highest weight representations of the $N = 2$ algebra with $c > 3$, which were obtained in [27] from the Kac determinant. For a given j, there is an infinite number of $N = 2$ representations labeled by $r \in \mathbb{Z}$. The number r is related to the eigenvalue m of J_0^3 in $\mathrm{SL}(2,\mathbb{R})$ in a way to be indicated below for each representation.

This construction allows to parameterize the quantum numbers of the $N = 2$ representations (the conformal dimension $h_{j,r}$ and U(1) charge $Q_{j,r}$ of the highest weights) in terms of $\mathrm{SL}(2,\mathbb{R})$ labels, and this will be very useful to make explicit the connection between $N = 2$ and $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ characters. Moreover, as we will show below, the $N = 2$ spectral flow (which will play a central role in our construction) just amounts to a shift in the quantum number r.

The relation between the characters of the $N = 2$ representations and the characters of the coset is similar to the embedding of the coset into $\mathrm{SL}(2,\mathbb{R})$ considered in section 2. Namely, the $N = 2$ characters can be expanded into terms with definite U(1) charge $Q = Q_{j,r} + n$ ($n \in \mathbb{Z}$). Then each term is the product of a U(1) character (generated by the modes of the R-current $J(z)$) with highest weight

$$\Delta = \frac{3Q^2}{2c} = \frac{k-2}{2k}\left(Q_{j,r} + n\right)^2 ,$$

times a coset character. We will work in the NS sector of the $N = 2$ algebra, which suffices for our purposes.

Continuous representations. From a $\mathrm{SL}(2,\mathbb{R})_k$ continuous representation with parameters $j = \frac{1}{2} + is, \alpha$ we obtain the $N = 2$ representations

$$h_{j,r} = \frac{\frac{1}{4} + s^2 + (\alpha + r)^2}{k-2} , \qquad Q_{j,r} = Q_m = \frac{2(r+\alpha)}{k-2} \tag{5.1}$$

with $r \in \mathbb{Z}$. The $\mathrm{SL}(2,\mathbb{R})$ state which gives rise to each representation has $J_0^3 = m = r + \alpha$, and we will use s, m to label the representations. There are no null states among the descendents [28, 29], so the $N = 2$ character is obtained from the free action of the modes

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$L_{-n}, J_{-n}, G^{\pm}_{-n+1/2}$ on the highest weight state. This yields

$$
\begin{aligned}
ch_c(s,m;\tau,\nu) &= \mathrm{Tr}\, q^{L_0-c/24} z^{J_0} = q^{-c/24} q^{h_{j,m}} z^{Q_m} \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}}z)(1+q^{n-\frac{1}{2}}z^{-1})}{(1-q^n)^2} \\
&= q^{-c/24} q^{1/8} q^{h_{j,m}} z^{Q_m} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}
\end{aligned} \tag{5.2}
$$

where

$$
\vartheta_3(\tau,\nu) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}z)(1+q^{n-\frac{1}{2}}z^{-1}) = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} z^n\,. \tag{5.3}
$$

Plugging the above identity into the character and using in the exponent of q the identity

$$
\frac{n^2}{2} + \frac{m^2}{k-2} = \frac{(n-m)^2}{k} + \frac{k-2}{2k}\left(\frac{2m}{k-2}+n\right)^2. \tag{5.4}
$$

we find the decomposition of the $N=2$ characters in terms of a U(1) boson and the coset characters as:

$$
ch_c(s,m;\tau,\nu) = \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} z^{Q_m+n} q^{\frac{k-2}{2k}(Q_m+n)^2} \hat{\lambda}_{1/2+is,n-m}(\tau) \tag{5.5}
$$

where $\hat{\lambda}_{1/2+is,n-m}$ are the characters of the continuous coset representation. We now consider the effect of applying the $N=2$ spectral flow mapping [30] by w integer units:

$$
\begin{aligned}
h &\to h + wQ + \frac{c}{6}w^2 \\
Q &\to Q + \frac{c}{3}w\,,
\end{aligned} \tag{5.6}
$$

to both expressions (5.2) and (5.5) of the continuous characters. We find:

$$
q^{\frac{c}{6}w^2}\, z^{\frac{c}{3}w}\, ch_c(s,m;\tau,\nu+w\tau) = ch_c(s,m+w;\tau,\nu)
$$

$$
= q^{-c/24} q^{1/8} q^{h_{j,m+w}} z^{Q_{m+w}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \tag{5.7}
$$

$$
= \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} z^{Q_m+\frac{wk}{k-2}+n} q^{\frac{k-2}{2k}(Q_m+\frac{wk}{k-2}+n)^2} \hat{\lambda}_{1/2+is,n-m}(\tau)\,, \tag{5.8}
$$

respectively, where in the second line we used $\vartheta_3(\tau,\eta+w\tau) = q^{-\frac{w^2}{2}} z^{-w} \vartheta_3(\tau,\eta)$. Two observations are in order. Firstly, we see explicitly that all the representations with fixed Casimir labeled by $j = 1/2 + is$ are mapped into each other under spectral flow. Secondly, we notice that, in the last expression, eq. (5.8), summing over spectral flow sectors that differ by $k-2$ units, corresponds to summing in the bosonic coset characters over sectors that differ by k units in the chiral momentum. We are thus led to introduce *extended $N=2$ characters* as follows:

$$
Ch_c(s,m) = \sum_{t\in\mathbb{Z}} ch_c(s,m+t(k-2);\tau,\nu)
$$

$$= q^{\frac{s^2}{k-2}} \Theta_{2m,k-2}\left(\tau, \frac{2\nu}{k-2}\right) \frac{\theta_3(\tau,\nu)}{\eta^3(\tau)} \tag{5.9}$$

$$= \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} \Theta_{n(k-2)+2m,\frac{k(k-2)}{2}}\left(\tau, \frac{2\nu}{k-2}\right) \hat{\lambda}_{1/2+is,n-m} \tag{5.10}$$

$$= \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} z^{Q_m+n} q^{\frac{k-2}{2k}(Q_m+n)^2} \hat{\Lambda}_{1/2+is,n-m} \,. \tag{5.11}$$

We get three equivalent expressions. Eq. (5.9) is obtained from summing (5.7) over the spectral flowed sectors. Eqs. (5.10)-(5.11) are obtained from two forms of summing (5.8). These expressions show the intimate connection between extended $N=2$ characters and extended bosonic coset characters. Note that the quantum number $2m$ now lies in the set $2m \in \mathbb{Z}_{2(k-2)}$.

Discrete representations. From a $\mathrm{SL}(2,\mathbb{R})_k$ discrete $\mathcal{D}_j^+$ representation with $0<j<k/2$, we obtain $N=2$ representations with the following values of $h_{j,r}$ and $Q_{j,r}$.

$$r \geq 0 \qquad h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k-2} \qquad Q_{j,r} = \frac{2(j+r)}{k-2} \tag{5.12}$$

$$r < 0 \qquad h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k-2} - r - \frac{1}{2} \qquad Q_{j,r} = \frac{2(j+r)}{k-2} - 1\,. \tag{5.13}$$

For $r \geq 0$ ($r<0$), these states are built out of a J^3 primary of $\mathrm{SL}(2,\mathbb{R})$ with $J_0^3 = m = j+r$ ($J_0^3 = m = j+r+1$). Note that the $\mathrm{SL}(2,\mathbb{R})$ state with $m=j$ gives rise to two $N=2$ representations, with $r=0,-1$. These correspond to chiral and anti-chiral primaries respectively, as follows from $h_{j,0} = Q_{j,0}/2$ in (5.12) and $h_{j,-1} = -Q_{j,-1}/2$ in (5.13).

Each representation has one non-degenerate null descendent at relative U(1) charge $+1$ (-1) for $r \geq 0$ ($r<0$) and level $r+1/2$ ($-r-1/2$) [28, 29].[11] For $r=0,-1$, the null states are clearly $G^+_{-1/2}|h_{j,0}\rangle$ and $G^-_{-1/2}|h_{j,-1}\rangle$, while for other values of r the null states are more complicated (see [29] for explicit expressions for the first levels).

The characters for every $r \in \mathbb{Z}$ are given by

$$\begin{aligned} ch_d(j,r;\tau,\nu) &= \mathrm{Tr}\, q^{L_0 - c/24} z^{J_0} \\ &= q^{-\frac{(j-1/2)^2}{k-2} + \frac{(j+r)^2}{k-2}} z^{\frac{2(j+r)}{k-2}} \frac{1}{1+zq^{1/2+r}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,, \qquad (5.14) \\ &= q^{-\frac{(j-1/2)^2}{k-2} + \frac{(j+r)^2}{k-2} - r - \frac{1}{2}} z^{\frac{2(j+r)}{k-2} - 1} \frac{1}{1+z^{-1}q^{-1/2-r}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,. \qquad (5.15) \end{aligned}$$

It is immediate to verify that also for these representations, the spectral flow by w units is equivalent to a shift $r \to r+w$. Both expressions (5.14) and (5.15) hold for every $r \in \mathbb{Z}$, but each form reflects the structure of the representation and the null states for a particular sign of r. For $r \geq 0$ the form (5.14) shows that we have the quantum numbers (5.12) of the primary state, and the factor $(1+zq^{1/2+r})^{-1}$ gets rid of the null state. An analogous statement holds for $r<0$ and the form (5.15).

[11] In the cases $j = 0, k/2$ the structure of the descendents is more involved. Modular properties of the extended $N=2$ character built from the state $j=m=0$, which is both $N=2$ chiral and anti-chiral, have been described in [26].

Another useful expression for the character is:

$$ch_d(j,r;\tau,\nu) = \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \frac{(q^s z)^{\frac{2j-1}{k-2}} q^{\frac{s^2}{k-2}} z^{\frac{2s}{k-2}}}{1+zq^s} \tag{5.16}$$

where $s = r + \frac{1}{2}$.

Using (5.3), we can expand (5.14) into

$$ch_d(j,r;\tau,\nu) = q^{-\frac{(j-1/2)^2}{k-2}+\frac{(j+r)^2}{k-2}} z^{\frac{2(j+r)}{k-2}} \eta(\tau)^{-3} \sum_{n\in\mathbb{Z}} \sum_{s=0}^{\infty} (-1)^s z^{s+n} q^{\frac{n^2}{2}+s(\frac{1}{2}+r)} . \tag{5.17}$$

By shifting $n \to n-s$ and using (5.4) with $m = j+r$, we obtain the decomposition

$$ch_d(j,r;\tau,\nu) = \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} z^{\frac{2(j+r)}{k-2}+n} q^{\frac{k-2}{2k}\left(\frac{2(j+r)}{k-2}+n\right)^2} \lambda_{j,r-n}(\tau) , \tag{5.18}$$

where $\lambda_{j,r-n}(\tau)$ are the characters of the discrete coset representations given in (2.8).

Summing over $k-2$ units of spectral flow, we again obtain three alternative expressions for the discrete extended $N=2$ characters:

$$\begin{aligned}
Ch_d(j,r) &= \sum_{n\in\mathbb{Z}} ch_d(j, r+n(k-2); \tau,\nu) \\
&= \sum_{n\in\mathbb{Z}} \frac{q^{-\frac{(j-1/2)^2}{k-2}+\frac{(j+r+n(k-2))^2}{k-2}} z^{\frac{2(j+r+n(k-2))}{k-2}}}{1+zq^{1/2+r+n(k-2)}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} && (5.19) \\
&= \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} \Theta_{n(k-2)+2(j+r),\frac{k(k-2)}{2}}\left(\tau, \frac{2\nu}{k-2}\right) \lambda_{j,r-n}(\tau) && (5.20) \\
&= \eta^{-1}(\tau) \sum_{n\in\mathbb{Z}} z^{\frac{2(j+r)}{k-2}+n} q^{\frac{k-2}{2k}\left(\frac{2(j+r)}{k-2}+n\right)^2} \Lambda_{j,r-n}(\tau) , && (5.21)
\end{aligned}$$

and we have chosen the fundamental domain $r \geq 0, r \in \mathbb{Z}_{k-2}$. The form (5.19) can be easily recast into (see (5.16))

$$\begin{aligned}
Ch_d(j,r;\tau,\nu) &= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \sum_{s\in r+\frac{1}{2}+\mathbb{Z}(k-2)} \frac{(q^s z)^{\frac{2j-1}{k-2}} q^{\frac{s^2}{k-2}} z^{\frac{2s}{k-2}}}{1+zq^s} && (5.22) \\
&= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \frac{1}{k-2} \sum_{r'\in\mathbb{Z}_{k-2}} e^{-2\pi i \frac{r'(r+1)}{k-2}} I\left(\frac{4}{k-2}, \frac{r'}{k-2} - \frac{1}{2}, \frac{2j-1}{k-2} - \frac{1}{2}; \tau, \nu\right),
\end{aligned}$$

where the function $I(k,a,b;\tau,\nu)$ is defined in eq. (A.1).

Relation to [26]. In this subsection, we show the relation between our extended $N=2$ characters with those introduced in [26]. We first recall that in [26], super-Liouville theory was studied as an $N=2$ superconformal theory with central charge $c = 3\hat{c} = 3+3\mathcal{Q}^2$ with $\mathcal{Q} = \sqrt{2K/N}$ where K and N are positive integers (with greatest common divisor equal to one).[12] To link that study to ours, we identify the central charges of the respective $N=2$ algebras. For integer level k, this leads to.[13]

$$\begin{aligned} c &= \frac{3k}{k-2} = 3 + \frac{6K}{N} \\ K &= 1 \\ N &= k-2\,. \end{aligned} \tag{5.23}$$

The characters themselves are matched by identifying the quantum numbers as follows. Our U(1) charge $Q_{j,r}$ of the highest weight is Q in [26]. For the continuous characters (5.9) our quantum numbers $\frac{s^2}{k-2}$ and $2m \in Z_{2(k-2)}$ correspond to $p^2/2$ and $j \in Z_{2NK}$ in eq.(2.25) of [26] ("massive characters"). For the discrete characters, our quantum numbers (j,r) correspond to $(s/2,r)$ in [26]. The identity between our discrete extended character (5.19) and eq.(2.19) of [26] ("massless characters"), can be seen by rewriting (5.19) as

$$Ch_d(j,r) = \sum_{n\in\mathbb{Z}} \frac{\left\{ zq^{(k-2)\left(n+\frac{2r+1}{2(k-2)}\right)}\right\}^{\frac{2j-1}{k-2}}}{1+zq^{(k-2)(n+\frac{2r+1}{2(k-2)})}} z^{2\left(n+\frac{2r+1}{2(k-2)}\right)} q^{(k-2)\left(n+\frac{2r+1}{2(k-2)}\right)^2} \frac{\theta_3(\tau,\nu)}{\eta^3(\tau)}\,. \tag{5.24}$$

Several useful properties of the extended characters are listed in [26]. We now turn to the next step in the program, which is to rederive their modular transformation properties. Finally, we will use the decomposition formulas to dissect the modular transformation rules.

6. The $N=2$ modular transformations

The modular transformation S of the continuous extended $N=2$ characters can be derived using the same techniques as we employed for the extended continuous coset characters. We define the formula for the (radial) momentum s in terms of the conformal dimension and U(1) R-charge: $s^2/(k-2) = h' - [(2m)^2 + 1/4](k-2)$, where the U(1) charge is: $Q = 2m/(k-2) = 2(\alpha+r)/(k-2)$. We then find [26]:

$$\begin{aligned} Ch_c\left(s,m;-\frac{1}{\tau},\frac{\nu}{\tau}\right) &= \frac{2}{k-2} e^{i\pi\frac{k}{k-2}\frac{\nu^2}{\tau}} \sum_{2m'\in Z_{2(k-2)}} e^{-\frac{4i\pi}{k-2}mm'} \times \\ &\quad \times \int_0^\infty ds' \cos\left(\frac{4\pi s s'}{k-2}\right) Ch_c(s',m';\tau,\nu)\,. \end{aligned} \tag{6.1}$$

To derive the modular transformation properties of the discrete extended $N=2$ characters, we will need a useful lemma given in appendix A.[14] Using this lemma it is possible to rederive the modular transformation properties as in [26]. We only use the restricted case

[12] Only in this subsection will N denote this positive number and does not refer to the number of supercurrents in the superconformal algebra.

[13] The $N=2$ techniques of [26] can also be used to treat the case of rational values of k.

[14] A proof of this lemma is given in [25] for square tori and real ν. We extend the proof to the generic case in appendix A. Note that this provides a firm backing of the results in e.g. [26, 31].

of relevance to us in the following. The extended discrete $N=2$ characters transform into discrete and continuous characters as follows:[15,16]

$$
\begin{aligned}
e^{-i\pi\frac{c\nu^2}{3\tau}} Ch_d\left(j,r;-\frac{1}{\tau},\frac{\nu}{\tau}\right) = & \frac{1}{k-2} \sum_{2m'\in Z_{2(k-2)}} e^{-\frac{2i\pi}{k-2}(2j+2r)m'} \int_0^\infty ds' \times \\
& \times \frac{\cosh\pi\left(s'\frac{k-4j}{k-2}+im'\right)}{\cosh\pi(s'+im')} Ch_c(s',m';\tau,\nu) + \\
& + \frac{i}{k-2} \sum_{r'\in Z_{k-2}} \sum_{2j'=2}^{k-2} e^{-2\pi i\frac{(2j+2r)(2j'+2r')-(2j-1)(2j'-1)}{2(k-2)}} \times \\
& Ch_d(j',r';\tau,\nu) + \frac{i}{2(k-2)} \sum_{r'\in Z_{k-2}} e^{-2\pi i\frac{(2j+2r)(2r'+1)}{2(k-2)}} \times \\
& \times \left[Ch_d\left(\frac{1}{2},r';\tau,\nu\right) - Ch_d\left(\frac{(k-1)}{2},r';\tau,\nu\right)\right]
\end{aligned} \tag{6.2}
$$

We thus recalled the method for the derivation of the modular transformation properties of the extended $N=2$ characters, and have strengthened the backbone of the proof by extending the lemma in appendix A to generic modular parameters. The modular transformation properties have also been clarified slightly, by labeling the characters by natural $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ labels. It is appropriate here to recall that the extended $N=2$ characters were introduced in [26] to have a discrete spectrum of R-charges in the right-hand side of the modular transformation of discrete characters, eq. (6.2).

Check for $N=2$. A useful check on the above modular transformation property for discrete characters is given by computing whether the square of the modular S-matrix is the charge conjugation matrix. We can perform this check by (artificially) distinguishing a discrete and a continuous sector, and by viewing the S-matrix as being upper triangular:

$$
S = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \tag{6.3}
$$

where A indicates the coefficients of the discrete characters contributing to the modular transform of discrete characters, etcetera. The technical details are recorded in appendix B. We find, after using the character identities:

$$
Ch_c\left(s=0,r+\frac{1}{2};\tau,\nu\right) = Ch_d\left(\frac{1}{2},r;\tau,\nu\right) + Ch_d\left(\frac{(k-1)}{2},r;\tau,\nu\right), \tag{6.4}
$$

[15] In the computation, the principal value prescription is implemented at the point where we split the momentum integral into negative and positive momenta. In recombining the two integrals, we have eliminated a possible pole contribution (which the principal value prescription in the appendix A tells us to do) at $s'=0$. The final expression can be shown to be regular at $s'=0$.

[16] Note that the "partial range" of quantum numbers appearing on the right hand side, discussed in [26], namely $1\le 2j'\le k-1$, coincides in the particular case under study with the full range [24] of extended discrete characters necessary to describe all discrete representations in the coset. It is no surprise that all relevant characters appear on the right hand side, as all characters in the unitary supersymmetric coset model will satisfy the lower bound on the conformal dimension associated to the partial range discussed in [26]. It is nevertheless striking that the precise bound on the discrete representation label is naturally reproduced from an analysis of generic discrete $N=2$ characters.

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$$
\begin{aligned}
Ch_d(j,r;\tau,\nu) &= Ch_d\left(\frac{k}{2}-j,-r-1;\tau,-\nu\right), \\
Ch_c(s,m;\tau,\nu) &= Ch_c(s,-m;\tau,-\nu)\,,
\end{aligned} \tag{6.5}
$$

that the matrix S^2 is indeed the charge conjugation matrix, which changes the sign of the U(1)-charge. In the appendix we show in detail how the split into continuous and discrete representations indeed turns out to be artificial, which can be expected on the basis of the splitting of the continuous $N=2$ representation with $s=0$ and odd U(1) R-charge into two discrete ones (see equation (6.4)). This is a faithful foreshadowing of what will happen in the computation for the coset.

7. The modular transformations decomposed

We now wish to plug the decomposition rule for the $N=2$ characters recalled in section 5 in terms of coset characters into the modular transformation rules which we rederived in section 6.

Continuous transformations revisited. We want to start with the simple case of rederiving the modular transformation rules for extended continuous coset characters using $N=2$ characters. The analysis of the modular transformation properties of the continuous characters of the coset goes as follows:

$$
\begin{aligned}
Ch_c\left(s,\alpha+r;-\frac{1}{\tau},\frac{\nu}{\tau}\right) &= \frac{1}{\sqrt{-i\tau}\eta(\tau)}\sum_n\sqrt{\frac{-i\tau}{k(k-2)}}e^{\pi i\frac{k}{k-2}\frac{\nu^2}{\tau}}\times \\
&\quad\times\sum_{m\in Z_{k(k-2)}}e^{-2i\pi\frac{m(n(k-2)+2(\alpha+r))}{k(k-2)}}\Theta_{m,\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\times \\
&\quad\times\hat{\lambda}_{1/2+is,n-(\alpha+r)}\left(-\frac{1}{\tau}\right)\times \\
&= \frac{1}{\eta(\tau)}e^{i\pi\frac{k\nu^2}{\tau(k-2)}}\frac{2}{k-2}\sum_{q\in Z_{2(k-2)}}e^{-2\pi iq(\alpha+r)/(k-2)}\times \\
&\quad\times\int_0^\infty ds'\cos\left(\frac{4\pi ss'}{k-2}\right)\times \\
&\quad\times\sum_n\Theta_{n(k-2)+q,\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\hat{\lambda}_{1/2+is',n-q/2}(\tau)
\end{aligned} \tag{7.1}
$$

where the first line follows from the use of the modular transformation properties of Θ-functions (3.3), and the second line by using the transformation property of the $N=2$ characters (6.1) (which we decomposed into coset characters using (5.10)). We should identify coefficients of truly linearly independent Θ-functions (of which there are $k(k-2)$ at level $k(k-2)/2$). To that end, we need to split the sums in the right hand side as $n'=k\tilde{n}+p$ where $n\in\mathbb{Z}$ and $p\in\mathbb{Z}_k$, while we split $q=(k-2)q'+t$, where $q'\in\mathbb{Z}_2$ and $t\in\mathbb{Z}_{k-2}$. We then shift $p+q'\to p$ and find:

$$
\sum_n e^{-2\pi imn/k}\hat{\lambda}_{1/2+is,n-(\alpha+r)}(\tilde{q}) = 2\sqrt{\frac{k}{k-2}}\sum_{\tilde{n}}\sum_{q'\in\mathbb{Z}_2}\int_0^\infty ds'\cos\left(\frac{4\pi ss'}{k-2}\right)\times
$$

$$\times e^{-2\pi it/k(\alpha+r)} e^{4\pi ip(\alpha+r)/k} \times$$
$$\times e^{-2\pi iq'(\alpha+r)} \hat{\lambda}_{1/2+is',k\tilde{n}+p-kq'/2-t/2}(q)\,. \qquad (7.2)$$

where we have identified: $m \equiv t \bmod k-2$ and $m-t \equiv (k-2)p$ where $p \in \mathbb{Z}_k$. We can read the obtained result as a modular transformation property of continuous coset characters. After performing a discrete Fourier transform on m (by summing over $m \in \mathbb{Z}_k$), we recognize on the left hand side the extended continuous characters we defined; on the right hand side the remaining constrained sum over $kq'+t-2p$ gives the sum over $2(\alpha'+r') \in \mathbb{Z}_{2k}$. We realize that this formula agrees exactly with the result (3.4) we derived in another manner previously. Thus we have rederived the transformation property of extended continuous characters of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset in an admittedly roundabout fashion, but we have learned useful techniques to be applied in the case of the extended discrete characters.

Modular transformations of discrete extended coset characters. Now we have made all the necessary preparations to calculate the modular transformation properties of extended discrete bosonic coset characters. We follow the same reasoning as we just did for the continuous representations. By applying on the one hand the modular transformation properties of Θ-functions (3.3), and on the other hand the modular transformation properties of $N=2$ characters (6.2), we obtain:

$$e^{-\pi i\frac{k}{k-2}\nu^2/\tau} Ch_d\left(j,r;-\frac{1}{\tau},\frac{\nu}{\tau}\right) =$$
$$= \frac{1}{\eta(\tau)}\frac{1}{\sqrt{k(k-2)}}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}_{k(k-2)}} e^{-2i\pi\frac{m(n(k-2)+2(j+r))}{k(k-2)}} \times$$
$$\times\, \Theta_{m,\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\hat{\lambda}_{j,-n+r}\left(-\frac{1}{\tau}\right)$$
$$= \frac{1}{\eta(\tau)}\frac{1}{k-2}\sum_{q\in\mathbb{Z}_{2(k-2)}} e^{-2\pi iq(j+r)/(k-2)}\int_0^\infty ds'\,\frac{\cosh\pi\left(s'\frac{k-4j}{k-2}+i\frac{q}{2}\right)}{\cosh\pi(s'+i\frac{q}{2})} \times$$
$$\times \sum_{n'\in\mathbb{Z}}\Theta_{n'(k-2)+q,\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\hat{\lambda}_{\frac{1}{2}+is',n'-\frac{q}{2}}(\tau) +$$
$$+ \frac{i}{k-2}\frac{1}{\eta(\tau)}\sum_{r'\in Z_{k-2}}\sum_{2j'=2}^{k-2} e^{-\frac{\pi i}{k-2}[(2j+2r)(2j'+2r')-(2j-1)(2j'-1)]}\times$$
$$\times\sum_{n'\in\mathbb{Z}}\Theta_{n'(k-2)+2j'+2r',\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\lambda_{j',-n'+r'}(\tau) +$$
$$+ \frac{i}{2(k-2)}\frac{1}{\eta(\tau)}\sum_{r'\in Z_{k-2}} e^{-\frac{\pi i}{k-2}(2j+2r)(2r'+1)}\times$$
$$\times\left\{\sum_{n'\in\mathbb{Z}}\Theta_{n'(k-2)+1+2r',\frac{k(k-2)}{2}}\left(\tau,\frac{2\nu}{k-2}\right)\lambda_{\frac{1}{2},-n'+r'}(\tau) -\right.$$

$$- \sum_{n' \in \mathbb{Z}} \Theta_{n'(k-2)+k-1+2r', \frac{k(k-2)}{2}} \left(\tau, \frac{2\nu}{k-2} \right) \lambda_{\frac{k-1}{2}, -n'+r'}(\tau) \Bigg\} . \tag{7.3}$$

As in the continuous case, we equate the two alternative forms of the modular transform. We then fix m in the first expression and equate coefficients of linearly independent Θ-functions. The identification of linearly independent contributions is tedious but straightforward. As in the case of continuous characters we then perform a discrete Fourier transform and find:

$$\begin{aligned} \Lambda_{j,r}\left(-\frac{1}{\tau} \right) &= \frac{1}{\sqrt{k(k-2)}} \sum_{2(\alpha'+r') \in \mathbb{Z}_{2k}} e^{\frac{4\pi i}{k}(j+r)(\alpha'+r')} \times \\ &\times \int_0^\infty ds' \frac{\cosh \pi \left(s' \frac{k-4j}{k-2} + i\alpha' \right)}{\cosh \pi (s' + i\alpha')} \hat{\Lambda}_{\frac{1}{2}+is', \alpha'+r'} + \\ &+ \frac{i}{\sqrt{k(k-2)}} \sum_{2j'=2}^{k-2} \sum_{r' \in Z_k} e^{-\frac{4\pi i}{k}(j+r)(j'+r')} e^{\frac{4\pi i}{k-2}(j-1/2)(j'-1/2)} \Lambda_{j',r'}(\tau) + \\ &+ \frac{i}{2\sqrt{k(k-2)}} \sum_{r' \in \mathbb{Z}_k} \left\{ e^{-\frac{4\pi i}{k}(j+r)(\frac{1}{2}+r')} \Lambda_{\frac{1}{2},r'}(\tau) - e^{-\frac{4\pi i}{k}(j+r)(-\frac{1}{2}+r')} \Lambda_{\frac{k-1}{2},r'}(\tau) \right\} \end{aligned} \tag{7.4}$$

This is the quantity we wanted to calculate, namely the modular transform of the extended discrete character. The most important and new qualitative aspect of this formula is the mixing between discrete and continuous representations. The method we used can be extended to at least all rational rational values of k, albeit with some technical refinements.

Check for coset. The modular S-matrix is again upper-block diagonal, when we list the representations as a column vector $(discrete, continuous)$. When we name the blocks in the S-matrix A, B and D for the left-upper, right-upper and right-lower block, we have already checked earlier (in section 3) that the matrix D squares to the charge conjugation matrix in the continuous sector. The calculation for S^2 is done in detail in the appendix. Using the character identity

$$\hat{\lambda}_{\frac{1}{2},\frac{1}{2}+r} = \lambda_{\frac{k-1}{2},-r} + \lambda_{\frac{1}{2},-r-1} \tag{7.5}$$

one can then show that the charge conjugation matrix exchanges (j, r) with $(\frac{k}{2}-j, -r)$ for all j in the discrete sector. Again, the split of the continuous representation with odd parity into two discrete representations turned out to be crucial in allowing for a contribution from the "continuous" sector to the discrete sector, thus making for the expected charge conjugation matrix.[17]

[17] Note that we have the character identity:

$$\lambda_{j,r} = \lambda_{k/2-j,-r} ,$$

which shows that charge conjugate representations have equal characters, as required.

8. Conclusions

The main result of our paper is the formula for the modular transformation matrix S for both extended continuous and extended discrete $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset characters. We were able to derive it by embedding the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ modules into modules of an enlarged $N = 2$ superconformal algebra. An interesting aspect of the modular transformation matrix is that, given the presence of discrete representations, it inevitably links those up with both discrete and continuous representations. This is a qualitatively new and important property of the characters of these representations.

It seems important to generalize these results to the case of generic rational and irrational values of the level k, and possibly, to the case of non-extended coset characters. For the latter , one could apply our method on the transformation rules for ordinary $N = 2$ characters (see e.g. [31]).

As we have argued, we also view these computations as non-rational analogues of the derivation of the modular transformation properties of string functions, and their connection to the $N = 2$ characters with $c < 3$. We have thus laid bare a non-trivial modular S-matrix of a non-rational conformal field theory, and its connection to $N = 2$ characters with $c > 3$. We believe that the features of this S-matrix that we uncovered have analogues in many other examples of more complicated non-rational conformal field theories.

At the heart of our paper lies an extension of the computation of [25] which is based on the technique of shifting an integration contour over momenta and picking up poles corresponding to discrete representations on the way over. This is highly reminiscent of techniques used in other analyses of non-rational conformal field theories (see e.g. [32, 24, 33, 16]).

Given our modular S-transformation matrix one can now wonder whether an analogue of the Verlinde-formula would hold for this non-rational conformal field theory and whether one can classify modular invariants (in particular in the discrete sector). One can also use the modular S-matrix to study possible boundary states [16] (even) more systematically (as has been done for the rational case). It would indeed be interesting to study the systematics of boundary conformal field theories that can be constructed in non-rational conformal field theories once the modular S-matrix is known.

Finally, our study should be a useful tool to analyze the conjectured [23] / proven [34, 35] duality between the supersymmetric coset and $N = 2$ Liouville.

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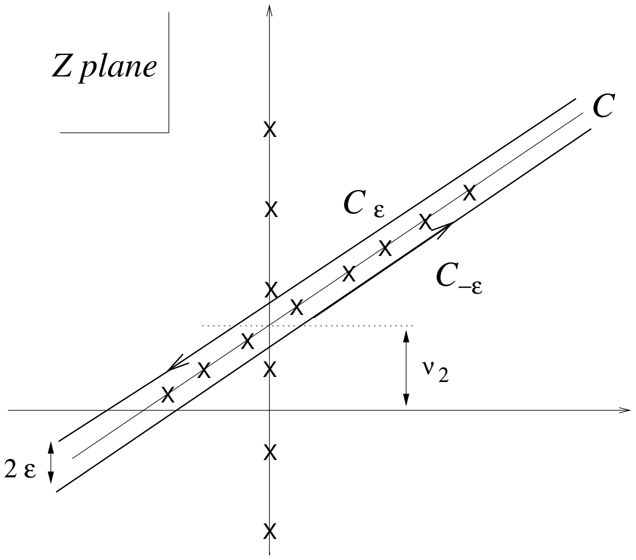

Figure 1: Choice of contour of integration (for $\tau_1 > 0$).

A. Lemma for $N = 2$ transformations

We follow [25] and derive the modular transformation properties of the following function:

$$I(k,a,b;\tau,\nu) = \sum_{r\in Z+\frac{1}{2}} e^{2\pi i(a+\frac{1}{2})(r+\frac{1}{2})} \frac{(zq^r)^{b+\frac{1}{2}}}{1+zq^r} z^{kr/2} q^{kr^2/4} \tag{A.1}$$

where $a, k \in \mathbb{R}$, $k > 0$ and $b \in \mathbb{C}$. We parameterize ν as: $\nu = \nu_1 - \tau\nu_2$, $\nu_{1,2} \in \mathbb{R}$. Our aim is to extend the proof of [25] — given for $\tau \in i\mathbb{R}$ and $\nu \in \mathbb{R}$ — to generic complex values. We would like to rewrite the S transform of this function as a contour integral in the following way:

$$\begin{aligned} B &= \frac{i}{\tau} e^{-\frac{ik\pi\nu^2}{2\tau}} I\left(k,a,b;-\frac{1}{\tau},\frac{\nu}{\tau}\right) \\ &= \frac{e^{i\pi a}}{2i\pi}\left[\int_{\mathcal{C}_{-\epsilon}} + \int_{\mathcal{C}_{+\epsilon}}\right] dZ \frac{\pi}{2} \frac{e^{-2\pi bZ - 2i\pi a(i\tau Z-\nu) + \frac{2i\pi\tau k}{4}Z^2}}{\cosh\pi Z\, \cos\pi(i\tau Z-\nu)} \end{aligned} \tag{A.2}$$

where the parameter ϵ is infinitesimal and positive. The contours encircles the line $\mathcal{C}:\ Z = iy/\tau + i\nu_2$, $y \in \mathbb{R}$ (figure 1). This equality is valid because the contour is such that we pick up all the poles of the integrand inside the contour of integration. These are the zeroes of $\cos\pi(i\tau Z - \nu)$ which occur at $i\tau Z - \nu = -y - \nu_1 \in \mathbb{Z} + 1/2$. At these poles, labeled by a half-integer r:

$$Z = \frac{i}{\tau}(r-\nu)$$

we find residues

$$(-1)^{r+\frac{1}{2}} \frac{i}{\tau} \frac{e^{2\pi i a(r+1/2) - 2i\pi b(r-\nu)/\tau} e^{-ik\pi(r-\nu)^2/(2\tau)}}{e^{i\pi(r-\nu)/\tau} + e^{-i\pi(r-\nu)/\tau}},$$

which leads to the equality quoted above. This equality is valid as long as there are no poles coming from the $\cosh \pi Z$ factor on the contour of integration. These special cases occur only for $\nu_2 \in \mathbb{Z} + 1/2$. We will not consider these special cases, which occur in the Ramond sector and are not crucial for our purposes. We now note that we have the expansions:

$$\frac{1}{2\cos\pi(i\tau Z - \nu)} = \sum_{n=0}^{\infty} (-1)^n e^{\pm(2n+1)\pi i(i\tau Z-\nu)} \tag{A.3}$$

whenever $|e^{\pm i2\pi(i\tau Z-\nu)}|$ is smaller than 1. This is true on the contours $\mathcal{C}_{\mp\epsilon}$. We then plug these expansions into the right hand side of eq. (A.2) and find:

$$B = \sum_{r\in Z+a+\frac{1}{2}} \frac{1}{2\pi i}\int_{\mathcal{C}} dZ \; J^b(r;Z) \tag{A.4}$$

where

$$J^b(r;Z) = i\pi e^{i\pi r} z^r \frac{e^{-2\pi bZ + 2\pi\tau rZ + 2i\pi\tau\frac{kZ^2}{4}}}{\cosh(\pi Z)},$$

since:

$$\begin{aligned} B = \; & \frac{1}{2\pi i}\int_{\mathcal{C}_{+\epsilon}} dZ\pi \sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)\pi i(i\tau Z-\nu)} \frac{e^{\pi ia - 2\pi bZ - 2\pi ia(i\tau Z-\nu) + 2i\pi\tau\frac{kZ^2}{4}}}{\cosh\pi Z} + \\ & + \frac{1}{2\pi i}\int_{\mathcal{C}_{-\epsilon}} dZ\pi \sum_{n=0}^{\infty}(-1)^n e^{+(2n+1)\pi i(i\tau Z-\nu)} \frac{e^{\pi ia - 2\pi bZ - 2\pi ia(i\tau Z-\nu) + 2i\pi\tau\frac{kZ^2}{4}}}{\cosh\pi Z} \end{aligned} \tag{A.5}$$

and we can identify $r = n + a + \frac{1}{2}$ and $r = -n + a - \frac{1}{2}$ in the first and second term respectively to obtain formula (A.4). When taking $\epsilon \to 0$ we also add a minus sign to the contour $\mathcal{C}_{+\epsilon}$ and switch its direction.

Now, in order to obtain the expression for the characters in the right hand side, we wish to *(i)* shift the $\mathcal{C}$ contour of integration by $\frac{2ir}{k} - i\nu_2$, for each term indexed by r and *(ii)* tilt it parallel to the real axis (figure 2). We will then eventually obtain the desired quadratic term $q^{r^2/k}$ which will match onto the quadratic term of continuous characters. The poles we pick up on this occasion arise from the zeroes of the factor $\cosh\pi Z$, at values of Z such that $Z = is$ with $s \in \mathbb{Z} + \frac{1}{2}$. When a pole falls exactly on the shifted contour we will count it with half its value and define the tilted integral over the real axis as a principal value (effectively accounting for the other half of the pole). We obtain two terms for B: one term, $J_1^{a,b}$, corresponds to the contribution of the poles. The second term, $J_2^{a,b}$, corresponds to the shifted contour. This last term is easily obtained:

$$\begin{aligned} J_2^{a,b} &= \sum_{r\in Z+a+\frac{1}{2}} \frac{1}{2\pi i} P\int_{\mathcal{C}+\frac{2ri}{k}-i\nu_2} dZ \; J^b(r;Z) \\ &= \sum_{r\in Z+a+\frac{1}{2}} e^{i\pi r} z^r q^{r^2/k} P\int_{\mathbb{R}} dx \; \frac{e^{-2\pi(b+\frac{1}{2})(x+2ri/k)} q^{kx^2/4}}{1+e^{-2\pi(x+2ri/k)}} \end{aligned} \tag{A.6}$$

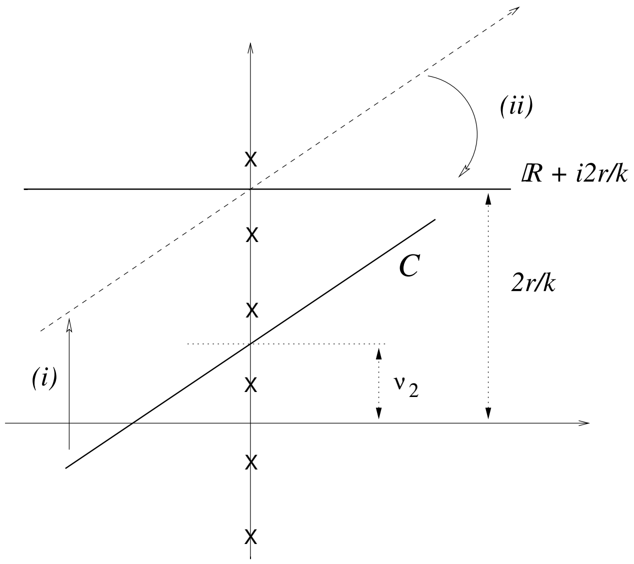

Figure 2: Change of contour of integration (case $r > \frac{k\nu_2}{2}$).

where in the last line the contour has been tilted to obtain an integral along the real axis. No poles are crossed during this deformation. Thus we obtain an integral over continuous characters of $N = 2$.

In the term $J_1^{a,b}$, the poles contribute positively for $r > \frac{k\nu_2}{2}$, and negatively for $r < \frac{k\nu_2}{2}$:

$$J_1^{a,b} = \left(\sum_{r > \frac{k\nu_2}{2}} \sum_{2r/k \geqslant s > \nu_2} - \sum_{r < \frac{k\nu_2}{2}} \sum_{2r/k \leqslant s < \nu_2} \right) \epsilon\left(s - \frac{2r}{k}\right) j^b(r, s) \tag{A.7}$$

where $j^b(r,s)$ is the residue of $J^b(r,Z)$ at the pole $Z = is$ (where $s \in \mathbb{Z}+\frac{1}{2}$) and $\epsilon(\gamma) = \frac{1}{2}$ for $\gamma = 0$ and $\epsilon(\gamma) = 1$ otherwise (which takes into account the principal value prescription).

In this expression for $J_1^{a,b}$, we wish to perform the sum over r for fixed s. Let us begin with the case $r > k\nu_2/2$. For every $s > \nu_2$ we need to determine those $r > k\nu_2/2$, $r \in Z + a + \frac{1}{2}$ such that $s \leqslant 2r/k$ (we assume $\nu_2 > 0$, but the opposite case is treated in the same way). There is always a number $\delta(a,s) \in [-\frac{1}{2}, \frac{1}{2})$ such that the pole in s is picked for the values of r given by $r = R + ks/2 + \delta(a,s)$ where $R = \frac{1}{2}, \frac{3}{2}, \ldots$

Then we consider the case $r < k\nu_2/2$. For every $s < \nu_2$, we have similarly to sum over $r = R + \frac{ks}{2} + \delta(a,s)$ (where now $\delta(a,s) \in (-\frac{1}{2}, \frac{1}{2}]$), and $R = -\frac{1}{2}, -\frac{3}{2}, \ldots$

We thus obtain that J_1 becomes (with r, s half-integer):

$$\begin{aligned} J_1^{a,b} &= \left(\sum_{s>\nu_2} \sum_{R>0} - \sum_{s<\nu_2} \sum_{R<0} \right) \epsilon(R + \delta(a,s)) j^b\left(\frac{ks}{2} + R + \delta(a,s), s\right) \\ &= \left(\sum_{s>\nu_2} \sum_{R>0} - \sum_{s<\nu_2} \sum_{R<0} \right) \epsilon(R + \delta(a,s))(-i) e^{2\pi i (\frac{k}{4} - b + \frac{1}{2})s} (e^{i\pi} z q^s)^{R+\delta(a,s)} z^{\frac{ks}{2}} q^{\frac{ks^2}{4}} \,. \end{aligned} \tag{A.8}$$

Noting that $|zq^s| = e^{-2\pi\tau_2(s-\nu_2)}$ and using the expansions:

$$\frac{1}{(1+x)} = \sum_{n=0}^{\infty}(-x)^n \qquad \text{for } |x|<1$$
$$\frac{1}{(1+x)} = -\sum_{n=1}^{\infty}(-x)^{-n} \qquad \text{for } |x|>1$$

we can sum on R. For the values of s such that $|\delta(a,s)| \neq \frac{1}{2}$ we get:

$$J_{1\alpha}^{a,b} = \sum_{\substack{s\in\mathbb{Z}+\frac{1}{2} \\ |\delta(a,s)|\neq\frac{1}{2}}} e^{2\pi i(k/4-b+\frac{1}{2})s} e^{i\pi\delta(a,s)} \frac{(zq^s)^{\frac{1}{2}+\delta(a,s)}}{1+zq^s} z^{ks/2} q^{ks^2/4} \tag{A.9}$$

while for both cases $\delta(a,s) = \pm\frac{1}{2}$, we get

$$J_{1\beta}^{a,b} = \frac{1}{2} \sum_{\substack{s\in\mathbb{Z}+\frac{1}{2} \\ |\delta(a,s)|=\frac{1}{2}}} e^{2\pi i(k/4-b+\frac{1}{2})s} e^{-i\pi/2} \frac{1-zq^s}{1+zq^s} z^{ks/2} q^{ks^2/4}. \tag{A.10}$$

so that

$$J_1^{a,b} = J_{1\alpha}^{a,b} + J_{1\beta}^{a,b}\,. \tag{A.11}$$

This is the lemma of [25] extended to generic τ and ν.

For comparison with [26], note that naming the function used in [26] to obtain the modular transformation of the discrete extended characters $I^{ES}(k,a,b;\tau,\nu)$, we have

$$I^{ES}(k,a,b;\tau,\nu) = e^{-i\pi a} I\left(2k, a-\frac{1}{2}, b-\frac{1}{2}; \tau,\nu\right).$$

B. Computing $S^2 = C$

Modular S-matrix of $N=2$. The details of the computations of the square of the S-matrices is given in this appendix. For the computation of S^2 for the $N=2$ extended characters, we simplify the computation slightly, and note on beforehand that the factors $e^{i\pi c/3\nu^2/\tau} \times e^{i\pi c/3(\nu/\tau)^2(-\tau)}$ cancel out in S^2, such that we don't need to take them along. Next, we distinguish the following block matrices in the upper-triangular modular matrix S:

$$\begin{aligned}
{A_{j,r}}^{j',r'} &= \frac{i}{k-2} e^{-\frac{\pi i}{k-2}((2j+2r)(2j'+2r')-(2j-1)(2j'-1))} \qquad \text{for } 2j' = 2,3,\ldots,k-2 \\
&= \frac{i}{2(k-2)} e^{-\frac{\pi i}{k-2}(2j+2r)(1+2r')} \qquad \text{for } 2j'=1 \\
&= -\frac{i}{2(k-2)} e^{-\frac{\pi i}{k-2}(2j+2r)(1+2r')} \qquad \text{for } 2j' = k-1 \\
{B_{j,r}}^{s',2m'} &= \frac{1}{k-2} e^{-\frac{2i\pi}{k-2}(2j+2r)m'} \frac{\cosh\pi\left(s'+im'+\frac{2s'(1-2j)}{k-2}\right)}{\cosh\pi(s'+im')}
\end{aligned}$$

$$D_{s,2m}{}^{s',2m'} = \frac{2}{k-2} e^{-\frac{4i\pi}{k-2}mm'} \cos\frac{4\pi ss'}{k-2}\,. \tag{B.1}$$

We can then compute the squares of A and D:

$$\begin{aligned}
A^2_{j,r}{}^{j'',r''} &= \delta_{j,k/2-j''}\delta_{r,-r''-1} \qquad \text{for } 2j'' = 2,3,\ldots,k-2\\
&= \frac{1}{2}\delta_{j,(k-1)/2}\delta_{r,-r''-1} - \frac{1}{2}\delta_{1/2,j}\delta_{r,-r''-1} \qquad \text{for } 2j'' = 1\\
&= \frac{1}{2}\delta_{1/2,j}\delta_{r,-r''-1} - \frac{1}{2}\delta_{(k-1)/2,j}\delta_{r,-r''-1} \qquad \text{for } 2j'' = k-1\\
D^2_{s,2m}{}^{s'',2m''} &= \delta^{k-2}_{2m+2m''}\delta(s-s'')\,.
\end{aligned} \tag{B.2}$$

Next, we calculate the off-diagonal block in the square of the modular matrix S:

$$\begin{aligned}
(B\cdot D)_{j,r}{}^{s'',2m''} &= \frac{1}{2(k-2)}\delta^{k-2}_{2j+2r+2m''}[p]\times\\
&\quad\times\Big\{\sin^{-1}\frac{\pi}{k-2}(2j-1+2is'') + \sin^{-1}\frac{\pi}{k-2}(2j-1-2is'') +\\
&\quad\quad + (-1)^p\Big(\cot\frac{\pi}{k-2}(2j-1+2is'') +\\
&\quad\quad\quad + \cot\frac{\pi}{k-2}(2j-1-2is'')\Big)\Big\} \qquad \text{for } 2j = 2,3,\ldots,k-2\\
&= \frac{1}{4}\sum_{\alpha'}\delta^{k-2}_{1+2r+2m''}[p](-1)^{2\alpha' p}\delta^+(s'') \qquad \text{for } 2j = 1\\
&= \frac{1}{4}\sum_{\alpha'}\delta^{k-2}_{1+2r+2m''}[p](-1)^{2\alpha'(p+1)}\delta^+(s'') \qquad \text{for } 2j = k-1
\end{aligned} \tag{B.3}$$

where δ^+ denotes the delta-function on the positive real half-line. We have also introduced the Kronecker symbol mod k; $\delta^k_m[p]$ means that $m = kp$. We also find:

$$\begin{aligned}
(A\cdot B)_{j,r}{}^{s'',2m''} &= \frac{1}{2(k-2)}\delta^{k-2}_{2j+2r+2m''}[p]\times \qquad\qquad \text{(B.4)}\\
&\quad\times\Big\{-\sin^{-1}\frac{\pi}{k-2}(2j-1+2is'') - \sin^{-1}\frac{\pi}{k-2}(2j-1-2is'') +\\
&\quad\quad + (-1)^p\Big(-\cot\frac{\pi}{k-2}(2j-1+2is'') - \cot\frac{\pi}{k-2}(2j-1-2is'')\Big)\Big\}\,.
\end{aligned}$$

Combining the two, we obtain for $A.B + B.D$:

$$\begin{aligned}
A.B + B.D = 0 &\qquad 2j = 2,3,\ldots,k-2\\
= \frac{1}{2}\delta_{-1-2r,2m''}\delta^+(s'') &\qquad 2j = 1\\
= \frac{1}{2}\delta_{-1-2r,2m''}\delta^+(s'') &\qquad 2j = k-1\,.
\end{aligned} \tag{B.5}$$

Combining this final result with the character identities in the bulk of the paper, one finds the charge conjugation matrix.

S^2 computation for the bosonic coset. Details of the computation of S^2 for the coset are as follows. We have:

$$
\begin{aligned}
A &= \frac{i}{\sqrt{k(k-2)}} e^{-\frac{\pi i}{k}(2j+2r)(2j'+2r')} e^{\frac{\pi i}{k-2}(2j-1)(2j'-1)} \qquad \text{for } 2j' = 2,\dots,k-2 \\
&= \frac{i}{2\sqrt{k(k-2)}} e^{-\frac{\pi i}{k}(2j+2r)(1+2r')} \qquad \text{for } 2j' = 1\dots \\
&= -\frac{i}{2\sqrt{k(k-2)}} e^{-\frac{\pi i}{k}(2j+2r)(-1+2r')} \qquad \text{for } 2j' = k-1\dots \\
D &= \frac{2}{\sqrt{k(k-2)}} e^{\frac{\pi i}{k}(2\alpha+2r)(2\alpha'+2r')} \cos\left(\frac{4\pi s s'}{k-2}\right) \\
B &= \frac{1}{\sqrt{k(k-2)}} e^{\frac{\pi i}{k}(2j+2r)(2\alpha'+2r')} \frac{\cosh \pi\left(s'\frac{k-4j}{k-2} + i\alpha'\right)}{\cosh \pi (s' + i\alpha')} \,.
\end{aligned} \tag{B.6}
$$

We can then calculate $A.B$ and $B.D$ respectively, which gives a matrix labeled by discrete and continuous indices (j,r) and $(s'', \alpha'' + r'')$. We find:

$$
\begin{aligned}
A.B &= \frac{1}{2(k-2)} \delta^k_{2j+2r-2\alpha''-2r''}[p] \times \\
&\quad \times \left\{ \frac{1 + (-1)^p \cos \frac{-i2\pi s'' + \pi(1-2j)}{k-2}}{\sin \frac{\pi}{k-2}(1-2j-2is'')} + \frac{1 + (-1)^p \cos \frac{i2\pi s'' + \pi(1-2j)}{k-2}}{\sin \frac{\pi}{k-2}(1-2j+2is'')} \right\} \\
B.D &= -\frac{1}{2(k-2)} \delta^k_{2j+2r+2\alpha''+2r''}[p] \times \\
&\quad \times \left\{ \frac{1 + (-1)^p \cos \frac{-i2\pi s'' + \pi(1-2j)}{k-2}}{\sin \frac{\pi}{k-2}(1-2j-2is'')} + \frac{1 + (-1)^p \cos \frac{i2\pi s'' + \pi(1-2j)}{k-2}}{\sin \frac{\pi}{k-2}(1-2j+2is'')} \right\} \\
&\qquad\qquad \text{for } 2j = 2,\dots,k-2 \\
&= \frac{1}{2} \delta^{2k}_{2r+1+2\alpha''+2r''} \delta^+(s'') \qquad \text{for } 2j = 1 \\
&= \frac{1}{2} \delta^{2k}_{2r-1+2\alpha''+2r''} \delta^+(s'') \qquad \text{for } 2j = k-1\,.
\end{aligned} \tag{B.7}
$$

Combining these results with the character identities in the bulk of our paper yields the result $S^2 = C$, where C is the matrix that exchanges the characters labeled (j,r) and $(\frac{k}{2} - j, -r)$ in the discrete sector and $(s, \alpha + r)$ and $(s, -(\alpha + r))$ in the continuous sector.

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